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LAMINAR FLOW OF A LIQUID IN A ROTATING CYLINDER IN A GRAVITATIONAL FIELD

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The laminar flow of a viscous liquid in a vertical rotating cylinder is studied.

We consider the axisymmetric flow of a viscous incompressible liquid in a semiinfinite rotating circular cylinder, where the axis of the cylinder is in the vertical direction (Fig. 1). The approximate dimensionless system of equations describing the flow of the liquid in cylindrical coordinates, together with the initial and boundary conditions, has the form [1]:

$$rU = \frac{\partial \Pi}{\partial r} - \frac{1}{\mathrm{Re}}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial rV}{\partial r}, \tag{1}$$

$$r\frac{\partial U}{\partial z} = \frac{1}{\mathrm{Re}}\left[3\frac{\partial U}{\partial r} + r\frac{\partial^2 U}{\partial r^2} - \frac{r}{2U}\left(\frac{\partial U}{\partial r}\right)^2\right], \tag{2}$$

$$\frac{\partial W}{\partial z} = -\frac{\partial \Pi}{\partial z} + \frac{1}{\mathrm{Fp}} + \frac{1}{\mathrm{Re}\, r}\frac{\partial}{\partial r} r \frac{\partial W}{\partial r}, \tag{3}$$

$$\frac{1}{r}\frac{\partial rV}{\partial r} + \frac{\partial W}{\partial z} = 0, \tag{4}$$

$$W|_{z=0} = 1, \quad V|_{z=0} = 0, \quad U|_{z=0} = 0, \quad \Pi|_{z=0} = P_0/\rho v_0^2, \tag{5}$$

$$W|_{r=1} = 0, \quad V|_{r=1} = 0, \quad U|_{r=1} = (\omega_0 R/v_0)^2 = \omega^2, \tag{6}$$

where

$$V = \frac{v_r}{v_0}; \quad U = \frac{1}{r^2}\left(\frac{v_\varphi}{v_0}\right)^2; \quad W = \frac{v_z}{v_0};$$

$$\Pi = P/\rho v_0^2; \quad \mathrm{Re} = \frac{R v_0}{\nu}; \quad \mathrm{Fp} = \frac{v_0^2}{gR}.$$

The system of equations (1) through (4) is obtained by replacing the derivatives $W\frac{\partial}{\partial z}$ by $\frac{\partial}{\partial z}$ with the assumption that the radial component V of the velocity is small in comparison with the axial W and rotational $\sqrt{r^2 U}$ components, and the derivative of the flow in the direction of the axis is much smaller than the derivatives with respect to the radial coordinate [2].

According to the method given in [1] for solving the system (1) through (6), we look for the function Π in the form:

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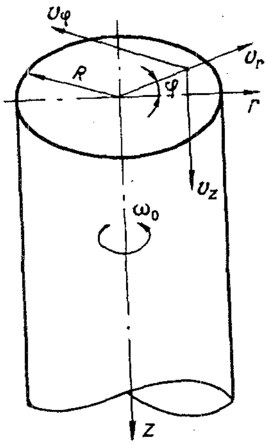

Fig. 1

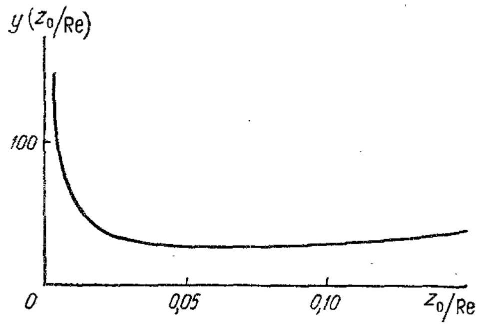

Fig. 2

Fig. 1. Coordinate system and directions of the velocity components v_r, v_φ, v_z.

Fig. 2. Dependence of the function $y(z_0/\mathrm{Re})$ on z_0/Re.

$$\Pi = q(z)\frac{r^2}{2} + Q(z) + z/\mathrm{Fp}. \tag{7}$$

We find from the condition (5)

$$q(0) = 0, \quad Q(0) = P_0/\rho v_0^2. \tag{8}$$

With no loss in generality we put $P_0 = 0$.

With the help of the representation of the function Π in the form of (7), Eq. (3) takes the form

$$\frac{\partial W}{\partial z} = \frac{1}{\mathrm{Re}\, r}\frac{\partial}{\partial r} r \frac{\partial W}{\partial r} - \frac{r^2}{2} q' - Q'. \tag{9}$$

Multiplying (9) by r and integrating the result with respect to r on [0, 1] with the help of the relation $\int_0^1 rW dr = \mathrm{const}$, we obtain

$$\frac{q'}{8} + \frac{Q'}{2} = \frac{1}{\mathrm{Re}}\left(\frac{\partial W}{\partial r}\right)\Big|_{r=1}. \tag{10}$$

Using the representation (7) and Eq. (4), we can write (1) as

$$U = q + \frac{1}{\mathrm{Re}\, r}\frac{\partial^2 W}{\partial r \partial z}.$$

Multiplying this expression by r and integrating the result with respect to r on [0, 1], we obtain

$$\int_0^1 rU dr = \frac{q}{2} + \frac{1}{\mathrm{Re}}\left[\left(\frac{\partial W}{\partial z}\right)\Big|_{r=1} - \left(\frac{\partial W}{\partial z}\right)\Big|_{r=1}\right]. \tag{11}$$

We linearize (2) by estimating the difference

$$\varepsilon = \frac{\partial^2 U}{\partial r^2} - \frac{1}{2U}\left(\frac{\partial U}{\partial r}\right)^2$$

and comparing it with the derivative $\partial U/\partial r$.

Let the distance from the wall of the pipe be $y = 1 - r$, then

$$\varepsilon = \frac{\partial^2 U}{\partial y^2} - \frac{1}{2U}\left(\frac{\partial U}{\partial y}\right)^2.$$

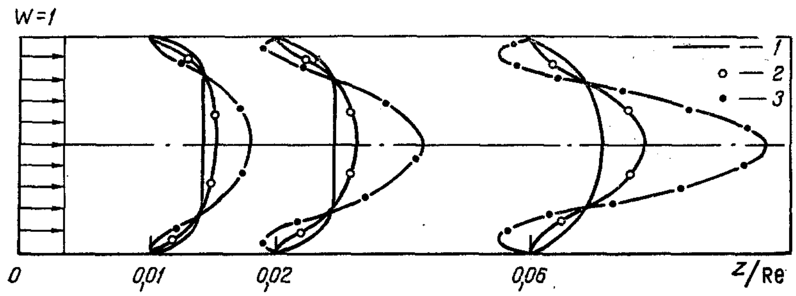

Fig. 3. Velocity profile W: 1) $\omega^2 = 0$; 2) 25; 3) 100.

Using the representation of the function U and the boundary conditions (6), the expansion of U in y near the wall of the pipe takes the form

$$U = \omega^2 + 2a\omega y + a^2y^2 + by^3 + cy^4 + \dots \tag{12}$$

Using (12), the quantity ε and the derivative $\partial U/\partial y$ can be written, to order y^2:

$$\varepsilon = (6b - 4a^3/\omega)\, y + (12c - 2a^4/\omega^2 - 6ab/\omega)\, y^2,$$
$$\frac{\partial U}{\partial y} = 2a\omega + 2a^2y + 3by^2. \tag{13}$$

Using (13) and the fact that the derivative $\partial U/\partial y$ and ε both have maximum values near $y = 0$, we see that the quantity ε can be neglected in Eq. (2).

Integrating (2) with respect to r on [0, 1], and using the estimates for the derivatives $\left(\frac{\partial^2 W}{\partial z^2}\right)\Big|_{r=1}$, $\left(\frac{\partial^2 W}{\partial z^2}\right)\Big|_{r=0}$, $\left(\frac{\partial^3 W}{\partial z \partial r^2}\right)\Big|_{r=0}$ at large Reynolds numbers given in [1], we obtain an equation for q(z)

$$\frac{1}{2}\, q' = \frac{3}{\mathrm{Re}}\, (\omega^2 - q). \tag{14}$$

The solution of (14), subject to the initial conditions (8), can be written as

$$q(z) = \omega^2\, [1 - \exp(-6z/\mathrm{Re})].$$

We look for the solution of (9) using the operational method. Let

$$\overline{W} = \int_0^\infty W \exp(-pz)\, dz.$$

We have the following equation for the transform $\overline{W}$

$$-1 + p\overline{W} = \frac{1}{\mathrm{Re}\, r} \frac{d}{dr}\, r\, \frac{d\overline{W}}{dr} - \frac{r^2}{2}\, \overline{q}' - \overline{Q}',$$
$$\overline{W}(1) = 0, \quad \left|\frac{d\overline{W}}{dr}\right|_{r=0} < +\infty. \tag{15}$$

Here $\overline{q}' = \dfrac{6\omega^2 (1/\mathrm{Re})}{p + 6(1/\mathrm{Re})}$, and the transform $\overline{Q}'$ satisfies the relation

$$\frac{\overline{q}'}{2} + \frac{\overline{Q}'}{8} = \frac{1}{\mathrm{Re}} \left(\frac{d\overline{W}}{dr}\right)\Big|_{r=1}. \tag{16}$$

Solving the system of equations (15) and (16), the final expression for $\overline{W}$ is

$$\overline{W} = -\frac{3(1/\mathrm{Re})\,\omega^2 - 2p - 12(1/\mathrm{Re})}{2p\,[p + 6(1/\mathrm{Re})]}\, \frac{I_0(\sqrt{p\,\mathrm{Re}}) - I_0(r\sqrt{p\,\mathrm{Re}})}{I_2(\sqrt{p\,\mathrm{Re}})} + \frac{3(1/\mathrm{Re})\,\omega^2(1 - r^2)}{p\,[p + 6(1/\mathrm{Re})]}. \tag{17}$$

Using the inversion theorem for the transform $\bar{W}$, and the residue theorem, we find for the velocity

$$W(z,\ r) = 2(1-r^2) - \frac{\omega^2}{4}\left[\frac{J_0(\sqrt{6}) - J_0(r\sqrt{6})}{J_2(\sqrt{6})} + 2(1-r^2)\right] \times$$
$$\times \exp(-6z/\mathrm{Re}) + 2\sum_{n=1}^{\infty} \frac{3\omega^2 + 2\mu_n^2 - 12}{\mu_n^2(6-\mu_n^2)}\left[1 - \frac{J_0(\mu_n r)}{J_0(\mu_n)}\right]\exp(-\mu_n^2 z/\mathrm{Re}). \tag{18}$$

The twisting of the flow due to the circular cylindrical pipe leads to back flow of the liquid near the wall of the pipe. Indeed, a numerical analysis of the expression for the derivative $\left(\frac{\partial W}{\partial r}\right)\Big|_{r=1}$ shows that for sufficiently large values of ω there exists a value of z for which $\left(\frac{\partial W}{\partial r}\right)\Big|_{r=1} = 0$. The distance z_0 for which $\left(\frac{\partial W}{\partial r}\right)\Big|_{r=1} = 0$, is found from the equation

$$\omega^2\left[\left(1 - \frac{\sqrt{6}\ J_1(\sqrt{6})}{4J_2(\sqrt{6})}\right)\exp(-6z_0/\mathrm{Re}) - 3\sum_{n=1}^{\infty}\frac{\exp(-\mu_n^2 z_0/\mathrm{Re})}{\mu_n^2 - 6}\right] - 4 - 2\sum_{n=1}^{\infty}\exp(-\mu_n^2 z_0/\mathrm{Re}) = 0. \tag{19}$$

This can be rewritten in the form

$$\omega^2 = \frac{4 + 2\sum_{n=1}^{\infty}\exp(-\mu_n^2 z_0/\mathrm{Re})}{0.28954\exp(-6z_0/\mathrm{Re}) - 3\sum_{n=1}^{\infty}\frac{\exp(-\mu_n^2 z_0/\mathrm{Re})}{\mu_n^2 - 6}}. \tag{20}$$

A graph of the function on the right hand side of (20) (denoted as $y(z_0/\mathrm{Re})$) is shown in Fig. 2.

The function $y(z_0/\mathrm{Re})$ has a minimum value of 25.91 at $z_0/\mathrm{Re} = 0.067$, and hence back flow does not occur for $\omega^2 < 25.91$. The boundary of the back flow for $\omega^2 > 25.91$ can be determined from Fig. 2. It is sufficient to find the abscissa of the point of intersection of the straight line $y = \omega^2$ with the function $y(z_0/\mathrm{Re})$.

The variation of the velocity profile W(z, r) is shown in Fig. 3.

We consider the behavior of the sum of the pressure and gravity forces acting on a particle of fluid near the wall of the pipe:

$$F_z = \left(-\frac{\partial \Pi}{\partial z} + \frac{1}{\mathrm{Fp}}\right)\Big|_{r=1}. \tag{21}$$

Using the representation (7) and the relation (10), the sum of the forces (21) can be written as

$$F_z = \frac{1}{\mathrm{Re}}\left\{8 + 4\sum_{n=1}^{\infty}\exp(-\mu_n^2 z/\mathrm{Re}) - \omega^2\left[2.07908\exp(-6z/\mathrm{Re}) - 6\sum_{n=1}^{\infty}\frac{\exp(-\mu_n^2 z/\mathrm{Re})}{\mu_n^2 - 6}\right]\right\}. \tag{22}$$

It follows from (22) that when

$$\omega^2 < \frac{4 + 2\sum_{n=1}^{\infty}\exp(-\mu_n^2 z/\mathrm{Re})}{1.03854\exp(-6z/\mathrm{Re}) - 3\sum_{n=1}^{\infty}\frac{\exp(-\mu_n^2 z/\mathrm{Re})}{\mu_n^2 - 6}} \tag{23}$$

we have $F_z > 0$ and the particle of fluid accelerates near the wall of the pipe up to the complete establishment of the velocity W corresponding to Poiseuille flow. The maximum value of ω^2 for which the inequality (23) holds for any value of z, is $\omega^2 = 5.77$. Obviously the function on the right hand side of (23) is less than the function y(z/Re) shown in Fig. 2 for any value of z.

Hence we conclude that when $\omega^2 > 25.91$, the force F_z changes sign at a certain distance z, and in this case a particle of fluid slows down and changes direction at $z = z_0$. For the region of twist parameters $5.77 < \omega^2 < 25.91$, the distance at which a fluid particle braked near the wall of the pipe is not large enough to lead to back flow near the wall.

The study of the behavior of F_z as a function of the distance z can be used in an experimental verification of the appearance of back flow near the wall.

NOTATION

r, z, coordinates of a point in the cylindrical coordinate system; v_r, v_φ, v_z, components of the velocity vector in cylindrical coordinates; P, pressure of the liquid; V, W, U, Π, dimensionless components of the velocity vector and pressure, respectively; P_0, v_0, pressure and velocity of the liquid entering the pipe; g, acceleration of gravity; ν, kinematic viscosity; R, radius of the pipe; ω_0, angular velocity of rotation of the pipe; Re, Fp, ω, Reynolds number, separation factor, and twist parameter, respectively; q(z), Q(z), dimensionless functions in the formula for Π; $\underline{\varepsilon}$, $\underline{a}$, $\underline{b}$, expansion parameters of the function U; p, parameter in the Laplace transform; $\bar{W}$, $\bar{q}'$, $\bar{Q}'$, Laplace transforms of the function W and the derivatives q' and Q'; I_0, I_2, Bessel functions of imaginary argument of order zero and two; J_0, J_2, Bessel functions of real argument or order zero and two; μ_n, zeros of the Bessel function J_2.

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DROPLET FORMATION FROM A JET OF ONE LIQUID ENTERING ANOTHER

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Wave theory has been used to derive expressions for the droplet sizes under various flow conditions.

When a jet of one liquid enters another but does not mix with it, waves are formed at the interface, which govern the break-up into droplets. If the jet is vertical, the break-up occurs in droplet, jet axisymmetric, jet bending, and spraying modes.

In droplet mode, the drops form at the end of the nozzle, which may be considered as a gravitational wave, length λ_g. As a drop forms, a capillary wave λ_σ forms at the surface, which moves over it towards the nozzle. If we neglect the efflux speed and assume that droplet formation ends when the capillary wave has traveled half the perimeter and reaches the axis, while the gravitational wave at the same time has traveled the nozzle radius, we have,

$$\pi D/(2)\, w_\sigma = d/(2)\, w_g. \tag{1}$$

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